NOTE OF ELEMENTARY ANALYSIS II

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1. Riemann Integrals

Notation 1.1. .

(i) : *All functions $f, g, h...$ are bounded real valued functions defined on $[a,b]$. And $m \le f \le M$.*
(ii) : *$\mathcal{P} : a = x_0 < x_1 < .... < x_n = b$ denotes a partition on $[a,b]$; $\Delta x_i = x_i - x_{i-1}$ and $\|\mathcal{P}\| = \max \Delta x_i$.*
(iii) : *$M_i(f,\mathcal{P}) := \sup\{f(x) : x \in [x_{i-1}, x_i\}$; $m_i(f,\mathcal{P}) := \inf\{f(x) : x \in [x_{i-1}, x_i\}$. And $\omega_i(f,\mathcal{P}) = M_i(f,\mathcal{P}) - m_i(f,\mathcal{P})$.*
(iv) : *$U(f,\mathcal{P}) := \sum M_i(f,\mathcal{P})\Delta x_i$; $L(f,\mathcal{P}) := \sum m_i(f,\mathcal{P})\Delta x_i$.*
(v) : *$\mathcal{R}(f,\mathcal{P},\{\xi_i\}) := \sum f(\xi_i)\Delta x_i$, where $\xi_i \in [x_{i-1}, x_i]$.*
(vi) : *$\mathcal{R}[a,b]$ is the class of all Riemann integral functions on $[a,b]$.*

Definition 1.2. *We say that the Riemann sum $\mathcal{R}(f,\mathcal{P},\{\xi_i\})$ converges to a number A as $\|\mathcal{P}\| \to 0$ if for any $\varepsilon > 0$, there is $\delta > 0$ such that*

$$|A - \mathcal{R}(f,\mathcal{P},\{\xi_i\})| < \varepsilon$$

for any $\xi_i \in [x_{i-1}, x_i]$ whenever $\|\mathcal{P}\| < \delta$.

Theorem 1.3. *$f \in \mathcal{R}[a,b]$ if and only if for any $\varepsilon > 0$, there is a partition $\mathcal{P}$ such that $U(f,\mathcal{P}) - L(f,\mathcal{P}) < \varepsilon$.*

Lemma 1.4. *$f \in \mathcal{R}[a,b]$ if and only if for any $\varepsilon > 0$, there is $\delta > 0$ such that $U(f,\mathcal{P}) - L(f,\mathcal{P}) < \varepsilon$ whenever $\|\mathcal{P}\| < \delta$.*

Proof. The converse follows from Theorem 1.3.
Assume that f is integrable over $[a,b]$. Let $\varepsilon > 0$. Then there is a partition $\mathcal{Q} : a = y_0 < ... < y_l = b$ on $[a,b]$ such that $U(f,Q) - L(f,Q) < \varepsilon$. Now take $0 < \delta < \varepsilon/l$. Suppose that $\mathcal{P} : a = x_0 < ... < x_n = b$ with $\|\mathcal{P}\| < \delta$. Then we have

$$U(f,\mathcal{P}) - L(f,\mathcal{P}) = I + II$$

where

$$I = \sum_{i : Q \cap (x_{i-1}, x_i) = \emptyset} \omega_i(f,\mathcal{P})\Delta x_i;$$

and

$$II = \sum_{i : Q \cap (x_{i-1}, x_i) \neq \emptyset} \omega_i(f,\mathcal{P})\Delta x_i$$

Notice that we have

$$I \le U(f,\mathcal{Q}) - L(f,\mathcal{Q}) < \varepsilon$$

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and

$$II \leq (M-m) \sum_{i:Q\cap(x_{i-1},x_i)\neq\emptyset} \Delta x_i \leq (M-m)\cdot l \cdot \frac{\varepsilon}{l} = (M-m)\varepsilon.$$

The proof is finished. ◻

Theorem 1.5. *$f \in \mathcal{R}[a,b]$ if and only if the Riemann sum $\mathcal{R}(f,\mathcal{P},\{\xi_i\})$ is convergent. In this case, $\mathcal{R}(f,\mathcal{P},\{\xi_i\})$ converges to $\int_a^b f(x)dx$ as $\|\mathcal{P}\| \to 0$.*

Proof. For the proof $(\Rightarrow)$: we first note that we always have

$$L(f,\mathcal{P}) \leq \mathcal{R}(f,\mathcal{P},\{\xi_i\}) \leq U(f,\mathcal{P})$$

and

$$L(f,\mathcal{P}) \leq \int_a^b f(x)dx \leq U(f,\mathcal{P})$$

for any $\xi_i \in [x_{i-1}, x_i]$ and for all partition $\mathcal{P}$.

Now let $\varepsilon > 0$. Lemma 1.4 gives $\delta > 0$ such that $U(f,\mathcal{P}) - L(f,\mathcal{P}) < \varepsilon$ as $\|\mathcal{P}\| < \delta$. Then we have

$$|\int_a^b f(x)dx - \mathcal{R}(f,\mathcal{P},\{\xi_i\})| < \varepsilon$$

as $\|\mathcal{P}\| < \delta$. The necessary part is proved and $\mathcal{R}(f,\mathcal{P},\{\xi_i\})$ converges to $\int_a^b f(x)dx$.
For $(\Leftarrow)$: there exists a number A such that for any $\varepsilon > 0$, there is $\delta > 0$, we have

$$A - \varepsilon < \mathcal{R}(f,\mathcal{P},\{\xi_i\}) < A + \varepsilon$$

for any partition $\mathcal{P}$ with $\|\mathcal{P}\| < \delta$ and $\xi_i \in [x_{i-1}, x_i]$.
Now fix a partition $\mathcal{P}$ with $\|\mathcal{P}\| < \delta$. Then for each $[x_{i-1}, x_i]$, choose $\xi_i \in [x_{i-1}, x_i]$ such that $M_i(f,\mathcal{P}) - \varepsilon \leq f(\xi_i)$. This implies that we have

$$U(f,\mathcal{P}) - \varepsilon(b-a) \leq \mathcal{R}(f,\mathcal{P},\{\xi_i\}) < A + \varepsilon.$$

So we have shown that for any $\varepsilon > 0$, there is a partition $\mathcal{P}$ such that

$$\overline{\int_a^b} f(x)dx \leq U(f,\mathcal{P}) \leq A + \varepsilon(1+b-a). \tag{1.1}$$

By considering $-f$, note that the Riemann sum of $-f$ will converge to $-A$. The inequality 1.1 will imply that for any $\varepsilon > 0$, there is a partition $\mathcal{P}$ such that

$$A - \varepsilon(1+b-a) \leq \underline{\int_a^b} f(x)dx \leq \overline{\int_a^b} f(x)dx \leq A + \varepsilon(1+b-a).$$

The proof is finished. ◻

Theorem 1.6. *Let $f \in \mathcal{R}[c,d]$ and let $\phi : [a,b] \longrightarrow [c,d]$ be a strictly increasing C^1 function with $f(a) = c$ and $f(b) = d$.*
Then $f \circ \phi \in \mathcal{R}[a,b]$, moreover, we have

$$\int_c^d f(x)dx = \int_a^b f(\phi(t))\phi'(t)dt.$$

Proof. Let $A = \int_c^d f(x)dx$. By Theorem 1.5, we need to show that for all $\varepsilon > 0$, there is $\delta > 0$ such that

$$|A - \sum f(\phi(\xi_k))\phi'(\xi_k)\triangle t_k| < \varepsilon$$

for all $\xi_k \in [t_{k-1}, t_k]$ whenever $\mathcal{Q} : a = t_0 < ... < t_m = b$ with $\|\mathcal{Q}\| < \delta$.
Now let $\varepsilon > 0$. Then by Lemma 1.4 and Theorem 1.5, there is $\delta_1 > 0$ such that

$$|A - \sum f(\eta_k)\triangle x_k| < \varepsilon \tag{1.2}$$

and

$$\sum \omega_k(f, \mathcal{P})\triangle x_k < \varepsilon \tag{1.3}$$

for all $\eta_k \in [x_{k-1}, x_k]$ whenever $\mathcal{P} : c = x_0 < ... < x_m = d$ with $\|\mathcal{P}\| < \delta_1$.
Now put $x = \phi(t)$ for $t \in [a, b]$.
Now since ϕ and ϕ' are continuous on $[a, b]$, there is $\delta > 0$ such that $|\phi(t) - \phi(t')| < \delta_1$ and $|\phi'(t) - \phi'(t')| < \varepsilon$ for all $t, t'\ in [a, b]$ with $|t - t'| < \delta$.
Now let $\mathcal{Q} : a = t_0 < ... < t_m = b$ with $\|\mathcal{Q}\| < \delta$. If we put $x_k = \phi(t_k)$, then $\mathcal{P} : c = x_0 < .... < x_m = d$ is a partition on $[c, d]$ with $\|\mathcal{P}\| < \delta_1$ because ϕ is strictly increasing.
Note that the Mean Value Theorem implies that for each $[t_{k-1}, t_k]$, there is $\xi_k^* \in (t_{k-1}, t_k)$ such that

$$\triangle x_k = \phi(t_k) - \phi(t_{k-1}) = \phi'(\xi_k^*)\triangle t_k.$$

This yields that

$$|\triangle x_k - \phi'(\xi_k)\triangle t_k| < \varepsilon \triangle t_k \tag{1.4}$$

for any $\xi_k \in [t_{k-1}, t_k]$ for all $k = 1, ..., m$ because of the choice of δ.
Now for any $\xi_k \in [t_{k-1}, t_k]$, we have

$$\begin{aligned} |A - \sum f(\phi(\xi_k))\phi'(\xi_k)\triangle t_k| \le\ & |A - \sum f(\phi(\xi_k^*))\phi'(\xi_k^*)\triangle t_k| \\ & + |\sum f(\phi(\xi_k^*))\phi'(\xi_k^*)\triangle t_k - \sum f(\phi(\xi_k^*))\phi'(\xi_k)\triangle t_k| \\ & + |\sum f(\phi(\xi_k^*))\phi'(\xi_k)\triangle t_k - \sum f(\phi(\xi_k))\phi'(\xi_k)\triangle t_k| \end{aligned} \tag{1.5}$$

Notice that inequality 1.2 implies that

$$|A - \sum f(\phi(\xi_k^*))\phi'(\xi_k^*)\triangle t_k| = |A - \sum f(\phi(\xi_k^*))\triangle x_k| < \varepsilon.$$

Also, since we have $|\phi'(\xi_k^*) - \phi'(\xi_k)| < \varepsilon$ for all $k = 1, .., m$, we have

$$|\sum f(\phi(\xi_k^*))\phi'(\xi_k^*)\triangle t_k - \sum f(\phi(\xi_k^*))\phi'(\xi_k)\triangle t_k| \le M(b - a)\varepsilon$$

where $|f(x)| \le M$ for all $x \in [c, d]$.
On the other hand, by using inequality 1.4 we have

$$|\phi'(\xi_k)\triangle t_k| \le \triangle x_k + \varepsilon \triangle t_k$$

for all k. This, together with inequality 1.3 imply that

$$\begin{aligned} & |\sum f(\phi(\xi_k^*))\phi'(\xi_k)\triangle t_k - \sum f(\phi(\xi_k))\phi'(\xi_k)\triangle t_k| \\ & \le \sum \omega_k(f, \mathcal{P})|\phi'(\xi_k)\triangle t_k| \ (\because \phi(\xi_k^*), \phi(\xi_k) \in [x_{k-1}, x_k]) \\ & \le \sum \omega_k(f, \mathcal{P})(\triangle x_k + \varepsilon \triangle t_k) \\ & \le \varepsilon + 2M(b - a)\varepsilon. \end{aligned}$$

Finally by inequality 1.5, we have

$$|A - \sum f(\phi(\xi_k))\phi'(\xi_k)\triangle t_k| \le \varepsilon + M(b-a)\varepsilon + \varepsilon + 2M(b-a)\varepsilon.$$

The proof is finished. $\square$

Example 1.7. *Define (*formally*) an improper integral* $\Gamma(s)$ *(called the* Γ*-function) as follows:*

$$\Gamma(s) := \int_0^\infty x^{s-1}e^{-x}dx$$

for $s \in \mathbb{R}$. *Then* $\Gamma(s)$ *is convergent if and only if* $s > 0$.

Proof. Put $I(s) := \int_0^1 x^{s-1}e^{-x}dx$ and $II(s) := \int_1^\infty x^{s-1}e^{-x}dx$. We first claim that the integral $II(s)$ is convergent for all $s \in \mathbb{R}$.
In fact, if we fix $s \in \mathbb{R}$, then we have

$$\lim_{x\to\infty} \frac{x^{s-1}}{e^{x/2}} = 0.$$

So there is $M > 1$ such that $\frac{x^{s-1}}{e^{x/2}} \le 1$ for all $x \ge M$. Thus we have

$$0 \le \int_M^\infty x^{s-1}e^{-x}dx \le \int_M^\infty e^{-x/2}dx < \infty.$$

Therefore we need to show that the integral $I(s)$ is convergent if and only if $s > 0$.
Note that for $0 < \eta < 1$, we have

$$0 \le \int_\eta^1 x^{s-1}e^{-x}dx \le \int_\eta^1 x^{s-1}dx = \begin{cases} \frac{1}{s}(1-\eta^s) & \text{if } s-1 \ne -1; \\ -\ln \eta & \text{otherwise .} \end{cases}$$

Thus the integral $I(s) = \lim_{\eta\to 0+} \int_\eta^1 x^{s-1}e^{-x}dx$ is convergent if $s > 0$.
Conversely, we also have

$$\int_\eta^1 x^{s-1}e^{-x}dx \ge e^{-1}\int_\eta^1 x^{s-1}dx = \begin{cases} \frac{e^{-1}}{s}(1-\eta^s) & \text{if } s-1 \ne -1; \\ -e^{-1}\ln \eta & \text{otherwise .} \end{cases}$$

So if $s \le 0$, then $\int_\eta^1 x^{s-1}e^{-x}dx$ is divergent as $\eta \to 0+$. The result follows. $\square$

References

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